

INEQUALITIES FOR A WEIGHTED INTEGRAL

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ABSTRACT. In the article, along with the procedure generalizing the mean value theorems for derivative from Rolle to Taylor, we generalize the Iyengar inequality for an integral to obtain some inequalities for weighted integrals involving bounded derivative of the integrands.

1. INTRODUCTION

The Iyengar inequality (see [7, 8, 10, 11]) involving bounded derivative of a function has been researched by many mathematicians from its appearance in 1938. For details please see the references [8, 10, 11]. Recently, it has aroused much interest again (see [1, 2, 3, 5, 9, 15]). The new results obtained in recent years are as follows.

In [4, 6, 12, 13, 14], using the mean value theorems for derivative, step by step, F. Qi and other coauthors generalized the Iyengar inequality to some inequalities for single and multiple integrals involving the bounded n -th derivatives of the integrands.

For given points $a = (a_1, \dots, a_m)$, $b = (b_1, \dots, b_m) \in \mathbb{R}^m$ and $a_i < b_i$, $i = 1, 2, \dots, m$, denote the m -rectangles by

$$(1.1) \quad Q_m = \prod_{i=1}^m [a_i, b_i], \quad Q_m(t) = \prod_{i=1}^m [a_i, c_i(t)],$$

where $c_i(t) = (1-t)a_i + tb_i$, $i = 1, 2, \dots, m$, $t \in [0, 1]$.

Let $\boldsymbol{\nu} = (\nu_1, \dots, \nu_m)$ be a multi-index, that is, $\nu_i = \text{integer} \geq 0$, with $|\boldsymbol{\nu}| = \sum_{i=1}^m \nu_i$. Let f be a function of several variables defined on Q_m , and its partial derivatives of $(n+1)$ -th order remain between the upper and lower bounds $M_{n+1}(\boldsymbol{\nu})$ and $N_{n+1}(\boldsymbol{\nu})$ as follows

$$(1.2) \quad N_{n+1}(\boldsymbol{\nu}) \leq D^{\boldsymbol{\nu}} f(x) \leq M_{n+1}(\boldsymbol{\nu}), \quad x \in Q_m,$$

where we define

$$(1.3) \quad D^{\boldsymbol{\nu}} f(x) = \partial^{n+1} f(x) \Big/ \prod_{i=1}^m \partial x_i^{\nu_i}.$$

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We introduce the following notations

$$(1.4) \quad A(\boldsymbol{\nu}) = \prod_{i=1}^m \frac{(b_i - a_i)^{\nu_i+1}}{(\nu_i + 1)!} \cdot M_{n+1}(\boldsymbol{\nu}),$$

$$(1.5) \quad B(\boldsymbol{\nu}, f(x)) = \prod_{i=1}^m \left(\frac{(b_i - a_i)^{\nu_i+1}}{(\nu_i + 1)!} \cdot \left(\frac{\partial}{\partial x_i} \right)^{\nu_i} \right) f(x),$$

$$(1.6) \quad C(\boldsymbol{\nu}) = \prod_{i=1}^m \frac{(b_i - a_i)^{\nu_i+1}}{(\nu_i + 1)!} \cdot N_{n+1}(\boldsymbol{\nu}).$$

Theorem 1.1 ([14]). *Let $f \in C^{n+1}(Q_m)$ and $N_{n+1}(\boldsymbol{\nu}) \leq D^\nu f(x) \leq M_{n+1}(\boldsymbol{\nu})$ hold for any $x \in Q_m$ and $|\boldsymbol{\nu}| = n + 1$, where $M_{n+1}(\boldsymbol{\nu})$ and $N_{n+1}(\boldsymbol{\nu})$ are constants depending on n and $\boldsymbol{\nu}$, then, for any $t \in [0, 1]$,*

(i) *when n is an even, we have*

$$(1.7) \quad \sum_{|\boldsymbol{\nu}|=n+1} C(\boldsymbol{\nu}) t^{m+n+1} + \sum_{|\boldsymbol{\nu}|=n+1} A(\boldsymbol{\nu}) T(\boldsymbol{\nu}, t) \\ \leq \int_{Q_m} f(x) dx - \sum_{k=0}^n \sum_{|\boldsymbol{\nu}|=k} B(\boldsymbol{\nu}, f(a)) t^{m+k} + \sum_{k=0}^n (-1)^k \sum_{|\boldsymbol{\nu}|=k} B(\boldsymbol{\nu}, f(b)) T(\boldsymbol{\nu}, t) \\ \leq \sum_{|\boldsymbol{\nu}|=n+1} A(\boldsymbol{\nu}) t^{m+n+1} + \sum_{|\boldsymbol{\nu}|=n+1} C(\boldsymbol{\nu}) T(\boldsymbol{\nu}, t);$$

(ii) *when n is an odd,*

$$(1.8) \quad \sum_{|\boldsymbol{\nu}|=n+1} C(\boldsymbol{\nu}) (t^{m+n+1} + T(\boldsymbol{\nu}, t)) \\ \leq \int_{Q_m} f(x) dx - \sum_{k=0}^n \sum_{|\boldsymbol{\nu}|=k} B(\boldsymbol{\nu}, f(a)) t^{m+k} + \sum_{k=0}^n (-1)^k \sum_{|\boldsymbol{\nu}|=k} B(\boldsymbol{\nu}, f(b)) T(\boldsymbol{\nu}, t) \\ \leq \sum_{|\boldsymbol{\nu}|=n+1} A(\boldsymbol{\nu}) (t^{m+n+1} + T(\boldsymbol{\nu}, t)).$$

Where $T(\boldsymbol{\nu}, t) = \prod_{i=1}^m \{1 - (1 - t)^{\nu_i+1}\} - 1$, $t \in [0, 1]$.

If we take $m = 1$ in Theorem 1.1, we could get the following

Corollary 1.1 ([13]). *Let $f(x)$ be a differentiable function of $C^{n+1}([a, b])$ satisfying $N \leq f^{(n+1)}(x) \leq M$. Denote*

$$(1.9) \quad S_n(u, v, w) = \sum_{k=1}^{n-1} \frac{(-1)^k}{k!} \cdot u^k f^{(k-1)}(v) + \frac{w}{n!} \cdot (-1)^n u^n,$$

$$(1.10) \quad \frac{\partial^k S_n}{\partial u^k} = S_n^{(k)}(u, v, w),$$

then, for any $t \in [a, b]$, when n is an odd we have

$$(1.11) \quad \sum_{i=0}^{n+2} \frac{(-1)^i}{i!} \left(S_{n+2}^{(i)}(a, a, N) - S_{n+2}^{(i)}(b, b, N) \right) t^i \leq \int_a^b f(x) dx$$

$$\leq \sum_{i=0}^{n+2} \frac{(-1)^i}{i!} \left(S_{n+2}^{(i)}(a, a, M) - S_{n+2}^{(i)}(b, b, M) \right) t^i;$$

when n is an even we have

$$(1.12) \quad \sum_{i=0}^{n+2} \frac{(-1)^i}{i!} \left(S_{n+2}^{(i)}(a, a, N) - S_{n+2}^{(i)}(b, b, M) \right) t^i \leq \int_a^b f(x) dx$$

$$\leq \sum_{i=0}^{n+2} \frac{(-1)^i}{i!} \left(S_{n+2}^{(i)}(a, a, M) - S_{n+2}^{(i)}(b, b, N) \right) t^i.$$

If we take $n = 1$ in Corollary 1.1 and choose suitable t , we obtain

Corollary 1.2 ([13]). *Let $f(x)$ be a 2-times differentiable function satisfying $N \leq f''(x) \leq M$, then*

$$(1.13) \quad \frac{N(b^3 - a^3)}{6} + \frac{\{f(a) - f(b) + bf'(b) - af'(a) + N(a^2 - b^2)/2\}^2}{2((a-b)N - f'(a) + f'(b))}$$

$$\leq \int_a^b f(x) dx - bf(b) + af(a) + \frac{b^2 f'(b) - a^2 f'(a)}{2}$$

$$\leq \frac{M(b^3 - a^3)}{6} + \frac{\{f(a) - f(b) + bf'(b) - af'(a) + M(a^2 - b^2)/2\}^2}{2((a-b)M - f'(a) + f'(b))}.$$

Proof. In inequality (1.11), taking $n = 1$ and

$$(1.14) \quad t = \frac{f(a) - f(b) - af'(a) + bf'(b) + N(a^2 - b^2)/2}{(a-b)N + f'(b) - f'(a)} \in [a, b],$$

the value of the left-hand side of (1.11) takes a maximum, and then the left-hand side inequality of (1.13) follows. Using the symmetry of inequality (1.11), the right-hand side inequality of (1.13) follows easily. ■

On taking $n = 0$ in Corollary 1.1 and choosing suitable t , we have

Corollary 1.3 ([12]). *Let $f(x)$ be continuous in $[a, b]$ and differentiable for $x \in (a, b)$. Suppose that $f(x)$ is not a constant, and that $N \leq f'(x) \leq M$ for $x \in (a, b)$. Then*

$$(1.15) \quad \frac{NM(b-a)^2 + 2(b-a)[Mf(a) - Nf(b)] + [f(a) - f(b)]^2}{2(M-N)} \leq \int_a^b f(x) dx$$

$$\leq -\frac{NM(b-a)^2 + 2(b-a)[Nf(a) - Mf(b)] + [f(a) - f(b)]^2}{2(M-N)}.$$

It is noted that the inequalities in (1.15), obtained by F. Qi in [12] using the mean value theorems for the derivative, and those obtained by R. P. Agarwal and S. S. Dragomir in [1], using a classical inequality due to Hayashi (see [8, 10, 11]), are the same. We remark that the proof of the inequalities in (1.15) by F. Qi in [12] is much simpler and more natural.

When substituting $N = -M$ in (1.15), the Iyengar inequality is recovered. As a special case of Corollary 1.3, we further obtain

Corollary 1.4 ([12]). *Let $f(x)$ be a continuous function in $[a, b]$ and differentiable for $x \in (a, b)$. Suppose that $f(a) = f(b) = 0$, and that $N \leq f'(x) \leq M$ for $x \in (a, b)$. If $f(x)$ is not identically zero, then $N < 0 < M$, and we have*

$$(1.16) \quad \left| \int_a^b f(x) dx \right| \leq \frac{NM(b-a)^2}{2(N-M)}.$$

Taking $N = -M = -\sup_{x \in (a,b)} |f'(x)|$ reduces inequality (1.16) to

$$(1.17) \quad \left| \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{4} \cdot \sup_{x \in (a,b)} |f'(x)|.$$

It is well-known that inequality (1.17) always appears in standard text books, for instance see [16], and in mathematical contests or examinations for undergraduate students. In [6], B.-N. Guo and F. Qi gave many proofs of the inequality (1.17).

In [3], using Hayashi's inequality, P. Cerone and S. S. Dragomir obtained some inequalities for weighted integrals to give trapezoidal type quadrature rules, which may be looked upon as a generalization of inequalities in (1.15) and the Iyengar inequality.

In this article, along with the procedure generalizing the mean value theorems for derivatives from Rolle to Taylor, we will generalize inequalities (1.7)–(1.17) for an integral to obtain some inequalities for weighted integrals involving the bounded derivative of the integrands.

2. MAIN RESULTS

In this section we will consider the weighted integral $\int_a^b w(x)f(x) dx$, where the weight $w(x)$ is a non-negative integrable function on $[a, b]$, $f(x)$ is a continuous function on $[a, b]$ with indicated derivatives in (a, b) . Define

$$(2.1) \quad h_{s,k}(t) = \int_s^t (x-s)^k w(x) dx, \quad s, t \in [a, b], \quad k \in \mathbb{N}.$$

Theorem 2.1. *Let $f(x)$ be a continuous function on $[a, b]$ and differentiable in (a, b) . Suppose that $f(a) = f(b) = 0$, and that $N = \inf_{x \in (a,b)} f'(x) > -\infty$, $M = \sup_{x \in (a,b)} f'(x) < \infty$. Let $w(x) \geq 0$ for all $x \in [a, b]$ and $h_{s,k}(t) = \int_s^t (x-s)^k w(x) dx$, $s, t \in [a, b]$, $k \in \mathbb{N}$. If $f(x)$ and $w(x)$ are not identically zero, then $N < 0 < M$, and*

$$(2.2) \quad Nh_{a,1}(t_1) - Mh_{b,1}(t_1) \leq \int_a^b w(x)f(x) dx \leq Mh_{a,1}(t_0) - Nh_{b,1}(t_0),$$

where $t_0 = \frac{aM-bN}{M-N} \in (a, b)$, $t_1 = \frac{aN-bM}{N-M} \in (a, b)$.

Proof. Firstly, $N < 0 < M$ is an immediate consequence of the Rolle's mean value theorem. The idea now is to apply the Lagrange mean value theorem again in order to estimate the weighted integral. Let θ be a parameter satisfying $\theta \in (a, b)$, and

write

$$\begin{aligned}\int_a^b w(x)f(x) dx &= \int_a^\theta w(x)[f(x) - f(a)] dx + \int_\theta^b w(x)[f(x) - f(b)] dx \\ &= \int_a^\theta w(x)(x-a)f'(\xi_1) dx + \int_\theta^b w(x)(x-b)f'(\xi_2) dx,\end{aligned}$$

where $a < \xi_1 < \theta < \xi_2 < b$. From $f'(\xi_1) \leq M$ and $f'(\xi_2) \geq N$, it now follows that

$$\begin{aligned}\int_a^b w(x)f(x) dx &\leq M \int_a^\theta (x-a)w(x) dx + N \int_\theta^b (x-b)w(x) dx \\ &= Mh_{a,1}(\theta) - Nh_{b,1}(\theta).\end{aligned}$$

Since

$$\begin{aligned}\frac{dh_{s,k}(t)}{dt} &= (t-s)^k w(t), \\ \frac{d(Mh_{a,1}(\theta) - Nh_{b,1}(\theta))}{d\theta} &= [(M-N)\theta + (bN - aM)]w(\theta),\end{aligned}$$

it is easy to see that the upper bound has its minimum value at the point $\theta = \frac{aM-bN}{M-N} \in (a, b)$.

Similarly, we have

$$\begin{aligned}\int_a^b w(x)f(x) dx &\geq N \int_a^\theta (x-a)w(x) dx + M \int_\theta^b (x-b)w(x) dx \\ &= Nh_{a,1}(\theta) - Mh_{b,1}(\theta),\end{aligned}$$

and, on maximising this with respect to θ , we find that the lower bound has its maximum value at the point $\theta = \frac{aN-bM}{N-M} \in (a, b)$.

The proof is complete. ■

Theorem 2.2. *Let $f(x)$ be a continuous function on $[a, b]$ and differentiable in (a, b) . Suppose that $f(x)$ is not a constant, and that $N = \inf_{x \in (a, b)} f'(x) > -\infty$, $M = \sup_{x \in (a, b)} f'(x) < \infty$. Let $w(x) \geq 0$ for all $x \in [a, b]$ and $h_{s,k}(t) = \int_s^t (x-s)^k w(x) dx$, $s, t \in [a, b]$, $k \in \mathbb{N}$. If $w(x)$ is not identically zero, then*

$$\begin{aligned}(2.3) \quad &\left[\frac{f(b) - f(a)}{b - a} - M \right] \cdot h_{b,1}(t_3) - \left[\frac{f(b) - f(a)}{b - a} - N \right] \cdot h_{a,1}(t_3) \\ &\leq \int_a^b w(x)f(x) dx - f(a)h_{a,0}(b) - \frac{f(b) - f(a)}{b - a} \cdot h_{a,1}(b) \\ &\leq \left[\frac{f(b) - f(a)}{b - a} - N \right] \cdot h_{b,1}(t_2) - \left[\frac{f(b) - f(a)}{b - a} - M \right] \cdot h_{a,1}(t_2)\end{aligned}$$

where $t_2 = \frac{aM-bN+f(b)-f(a)}{M-N} \in (a, b)$, $t_3 = \frac{aN-bM+f(b)-f(a)}{N-M} \in (a, b)$.

Proof. For $x \in [a, b]$ we set

$$\phi(x) = [f(x) - f(a)](b-a) - [f(b) - f(a)](x-a),$$

so that $\phi(a) = \phi(b) = 0$. We also have

$$\phi'(x) = (b-a)f'(x) - f(b) + f(a),$$

and hence

$$(b-a)N - f(b) + f(a) \leq \phi'(x) \leq (b-a)M - f(b) + f(a).$$

The required result (2.3) now follows from Theorem 2.1 applied to $\phi(x)$, on noting that

$$\begin{aligned} \int_a^b w(x)\phi(x) dx &= (b-a) \int_a^b w(x)[f(x) - f(a)] dx - [f(b) - f(a)] \int_a^b (x-a)w(x) dx \\ &= (b-a) \left[\int_a^b w(x)f(x) dx - f(a)h_{a,0}(b) \right] - [f(b) - f(a)]h_{a,1}(b). \end{aligned}$$

The proof is complete. ■

Theorem 2.3. *Let $f(x)$ be a differentiable function in $C^{n-1}([a, b])$ with the n -th derivative $f^{(n)}(x)$ in (a, b) such that $N \leq f^{(n)}(x) \leq M$ for all $x \in (a, b)$. Let $w(x) \geq 0$ for $x \in [a, b]$ and $h_{s,k}(t) = \int_s^t (x-s)^k w(x) dx$, $s, t \in [a, b]$, $k \in \mathbb{N}$. If $w(x)$ is not identically zero, then, for any $t \in (a, b)$, when n is an odd, we have*

$$\begin{aligned} (2.4) \quad & \frac{Nh_{a,n}(t) - Mh_{b,n}(t)}{n!} \\ & \leq \int_a^b w(x)f(x) dx + \sum_{i=0}^{n-1} \frac{f^{(i)}(b)h_{b,i}(t) - f^{(i)}(a)h_{a,i}(t)}{i!} \\ & \leq \frac{Mh_{a,n}(t) - Nh_{b,n}(t)}{n!}; \end{aligned}$$

when n is an even we have

$$\begin{aligned} (2.5) \quad & \frac{N(h_{a,n}(t) - h_{b,n}(t))}{n!} \\ & \leq \int_a^b w(x)f(x) dx + \sum_{i=0}^{n-1} \frac{f^{(i)}(b)h_{b,i}(t) - f^{(i)}(a)h_{a,i}(t)}{i!} \\ & \leq \frac{M(h_{a,n}(t) - h_{b,n}(t))}{n!}. \end{aligned}$$

Proof. Let t be a parameter such that $a < t < b$, and write

$$(2.6) \quad \int_a^b w(x)f(x) dx = \int_a^t w(x)f(x) dx + \int_t^b w(x)f(x) dx.$$

The well-known Taylor's formula states that

$$(2.7) \quad f(x) = \sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{i!} (x-a)^i + \frac{f^{(n)}(\xi)}{n!} (x-a)^n, \quad \xi \in (a, x),$$

$$(2.8) \quad f(x) = \sum_{i=0}^{n-1} \frac{f^{(i)}(b)}{i!} (x-b)^i + \frac{f^{(n)}(\eta)}{n!} (x-b)^n, \quad \eta \in (x, b).$$

Integrating on both sides of (2.7) over $[a, t]$ yields

$$(2.9) \quad \int_a^t w(x)f(x) dx = \sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{i!} h_{a,i}(t) + \frac{f^{(n)}(\xi)}{n!} h_{a,n}(t).$$

Since $N \leq f^{(n)}(x) \leq M$ for $x \in (a, b)$, then

$$(2.10) \quad \frac{Nh_{a,n}(t)}{n!} \leq \int_a^t w(x)f(x) dx - \sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{i!} h_{a,i}(t) \leq \frac{Mh_{a,n}(t)}{n!}.$$

Integrating on both sides of (2.8) over $[t, b]$ leads to

$$(2.11) \quad \int_t^b w(x)f(x) dx = - \sum_{i=0}^{n-1} \frac{f^{(i)}(b)}{i!} h_{b,i}(t) - \frac{f^{(n)}(\eta)}{n!} h_{b,n}(t).$$

When n is even, from (2.11), it follows that

$$(2.12) \quad -\frac{Nh_{b,n}(t)}{n!} \leq \int_t^b w(x)f(x) dx + \sum_{i=0}^{n-1} \frac{f^{(i)}(b)h_{b,i}(t)}{i!} \leq -\frac{Mh_{b,n}(t)}{n!};$$

when n is odd, the reversed inequalities of (2.12) hold.

Substituting of inequalities in (2.10) and (2.12) into (2.6) leads to the inequalities in (2.4) and (2.5) respectively. The proof is complete. ■

Remark 2.1. If we take $w(x) = 1$ for all $x \in [a, b]$ in Theorems 2.1, 2.2 and 2.3, we can obtain Corollaries 1.4, 1.3 and 1.1 respectively, and then the Iyengar inequality (1.15) is recovered again.

REFERENCES

- [1] R. P. Agarwal and S. S. Dragomir, An application of Hayashi's inequality for differentiable functions, *Computers Math. Appl.* **32** (1996), no. 6, 95–99.
- [2] P. Cerone and S. S. Dragomir, On a weighted generalization of Iyengar type inequalities involving bounded first derivative, *RGMIA Research Report Collection* **2** (1999), no. 2, 147–157. <http://rgmia.vu.edu.au>.
- [3] P. Cerone and S. S. Dragomir, Lobatto type Quadrature rules for functions with bounded derivative, *RGMIA Research Report Collection* **2** (1999), no. 2, 133–146. <http://rgmia.vu.edu.au>.
- [4] L.-H. Cui and B.-N. Guo, On proofs of an integral inequality and its generalizations, *Journal of Zhengzhou Grain College*, **17** (1996), Supplement, 152–154 and 158. (Chinese)
- [5] A. M. Fink, Bounds on the deviation of a function from its averages, *Czech. Math. Journal* **42** (1992), no. 117, 289–310.
- [6] B.-N. Guo and Feng Qi, Proofs of an integral inequality, *Mathematics and Informatics Quarterly* **7** (1997), no. 4, 182–184.
- [7] K. S. K. Iyengar, Note on an inequality, *Math. Student* **6** (1938), 75–76.
- [8] J.-C. Kuang, *Applied Inequalities (Changyong Budengshi)*, 2nd edition, Hunan Education Press, Changsha, China, 1993, Chapter 8, page 584. (Chinese)
- [9] G. V. Milovanović and J. E. Pečarić, Some considerations on Iyengar's inequality and some related applications, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No.* **544–576** (1976), 166–170.
- [10] D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, 1970, page 297.
- [11] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Inequalities Involving Functions and Their Integrals and Derivatives*, Chapter XV, Kluwer Academic Publishers, 1991.
- [12] F. Qi, Inequalities for an integral, *The Mathematical Gazette* **80** (1996), no. 488, 376–377.
- [13] F. Qi, Further generalizations of inequalities for an integral, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat.* **8** (1997), 79–83.
- [14] F. Qi, Inequalities for a multiple integral, *Acta Mathematica Hungarica* **84** (1999), no. 1-2, 19–26.
- [15] P. M. Vasić and G. V. Milonović, On an inequality of Iyengar, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No.* **544–576** (1976), 18–24.
- [16] L.-Z. Xu and X.-H. Wang, *Methods of Mathematical Analysis and Selected Examples (Shuxue Fenxi Zhong De Fangfa He Liti Xuanjiang)*, Revised Edition, Higher Education Press, Beijing, China, 1984. (Chinese)

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